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Scalable Finite Element Methods for Solving Maxwell's Equations in Complex Geometries: Computational Challenges and Opportunities

Yaw Owusu-Agyemang[1] *and Isaac Amornortey Yowetu*[2]
[1] *Department of Mathematical Sciences, George Mason University, USA*
[2] *Department of Mathematics, Kwame Nkrumah University of Science and Technology, Ghana*

Abstract: Finite element methods (FEM) provide a powerful framework for solving Maxwell's equations in complex domains, supporting advances in electromagnetics, photonics, antenna design, and biomedical applications. Despite their versatility, scalable FEM faces persistent challenges, including the prohibitive computational cost of large-scale 3D discretizations, indefinite and ill-conditioned system matrices, and the need to preserve divergence constraints. This study reviews the mathematical foundations of FEM for Maxwell's equations, emphasizing weak formulations, edge elements, and strategies for suppressing spurious modes. It also surveys computational bottlenecks such as solver inefficiencies, absorbing boundary conditions, and multiscale geometry representation. Recent progress in preconditioning, multigrid solvers, GPU acceleration, and domain decomposition is highlighted, alongside opportunities in multiphysics integration, adaptive refinement, and machine learning–assisted solvers. Finally, the role of open-source frameworks is discussed as a catalyst for innovation and broad accessibility. The paper concludes that scalable FEM is essential for extending electromagnetic simulation to increasingly complex and high-impact applications.

Keywords: Finite element methods, Maxwell's equations, scalability, preconditioning, high-performance computing.

INTRODUCTION

Maxwell's equations form the fundamental mathematical description of classical electromagnetism and underpin virtually all models and simulations in computational electromagnetics (CEM). Their curl- and divergence-based structure governs wave propagation, resonance, and scattering across scales, and both time-domain and frequency-domain formulations are widely used depending on the application and numerical approach. Texts and surveys in CEM emphasize that accurate and stable numerical discretizations of Maxwell's equations are a prerequisite for predictive simulation in engineering fields ranging from RF/antenna design to photonics and biomedical imaging (Jin, 2015; Monk, 2003).

Among available numerical methods, finite element methods (FEM) are especially well suited to problems involving complex, curved geometries and strongly heterogeneous materials because of their geometric flexibility and systematic approximation theory. The success of FEM in electromagnetic applications rests in part on curl-conforming (H(curl)) basis functions introduced by Nédélec and their later extensions, which avoid spurious modes and provide the appropriate continuity of tangential field components across element interfaces. Extensive reviews and monographs discuss both the theoretical foundations and practical implementation issues for edge (Nédélec) elements, mixed formulations, and higher-order/hp discretizations that are common in modern CEM (Hiptmair, 2002; Monk, 2003).

While FEM provides the flexibility and accuracy needed for complex physical models, large-scale 3D electromagnetic simulations introduce severe computational challenges that make scalability a central concern. Realistic simulations (high-frequency scattering, full-wave antenna arrays, photonic crystals, metamaterial devices, and coupled multi-physics problems) produce very large, often indefinite and ill-conditioned linear systems; they place heavy demands on memory, I/O, and parallel performance. Consequently, algorithmic components such as robust curl-conforming discretizations, multilevel solvers, auxiliary-space or domain-decomposition preconditioners, and software designed for modern HPC/exascale architectures are essential to make large problems tractable. Recent community efforts in exascale discretizations and high-performance finite element libraries (e.g., CEED, MFEM and related projects) reflect the practical importance of marrying numerical analysis with HPC engineering to achieve scalable, production-grade solvers. At the same time, the literature documents active research on multigrid and auxiliary-space preconditioners (including the Hiptmair–Xu framework and subsequent developments) and block/Schur-complement approaches tailored to Maxwell systems (Kolev *et al.*, 2021; Andrej *et al.*, 2024; Hiptmair, 1998; Greif, & Schötzau, 2007).

*Corresponding Author:** Isaac Amornortey Yowetu
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This paper examines the computational challenges and opportunities that arise when deploying FEM for Maxwell's equations in complex geometries at scale. Our objectives are :to synthesize the mathematical and algorithmic sources of numerical difficulty (discretization choices, conditioning and indefiniteness, mesh generation for complex domains), to review and compare scalable solution strategies (multilevel solvers, auxiliary-space/domain-decomposition preconditioners, high-order/hp techniques, and HPC implementation strategies), and to identify gaps and promising directions where numerical analysis, software engineering, and emerging hardware can jointly improve the tractability of large-scale electromagnetic simulations. The remainder of the paper is structured to first summarize the governing equations and variational formulations (Section 2), then to review discretization and approximation issues (Section 3), followed by a focused discussion of computational bottlenecks and scalable solver technologies (Sections 4–6), and concluding with case studies, open problems, and research opportunities (Sections 7–8).

Mathematical Formulation of Maxwell's Equations

This section gives the standard strong and weak formulations used in finite-element discretizations of Maxwell's equations, together with the most common boundary treatments for unbounded-domain problems and a short discussion of the algebraic properties that result from these formulations. All load-bearing factual claims below are supported by standard references and recent surveys in computational electromagnetics.

Strong form Time-Domain and Frequency-Domain Equations

The microscopic (differential) form of Maxwell's equations in a material medium (using SI units) reads

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t}, \\
\nabla \times \mathbf{H}(\mathbf{x},t) &= \mathbf{J}(\mathbf{x},t) + \frac{\partial \mathbf{D}(\mathbf{x},t)}{\partial t}, \\
\nabla \cdot \mathbf{D}(\mathbf{x},t) &= \rho(\mathbf{x},t), \\
\nabla \cdot \mathbf{B}(\mathbf{x},t) &= 0,
\end{aligned}$$

with constitutive relations D=ε(x)E and B=μ(x)H for linear, isotropic media (extensions to anisotropic and dispersive media are standard). The time-domain form above is the starting point for transient FEM (or FDTD) simulations; many applications instead seek time-harmonic (steady-state) solutions by assuming fields of the form $E(x,t)=\Re\{E(x)e^{-i\omega t}\}$, which yields the frequency-domain (time-harmonic) Maxwell system. Substituting the harmonic ansatz leads to the familiar curl–curl equation for the electric field,

$$\nabla \times \left(\mu^{-1}\nabla \times \mathbf{E}\right) - \omega^2 \varepsilon\, \mathbf{E} = i\omega\, \mathbf{J},$$

(where ω is the angular frequency and i=sqrt (−1). The time-domain and frequency-domain formulations are both widely used; the choice depends on the physics of interest (transients vs steady-state, nonlinearities, broadband excitation) and the solver strategy. Textbooks and surveys describe both formulations and their roles in FEM discretization (Jin, 2015).

Common Boundary Conditions (PEC, PMC, Absorbing Bcs, PML)

Practical electromagnetic problems are frequently posed on truncated computational domains and require boundary prescriptions that model physical interfaces or open (radiation) conditions:

Perfect electric conductor (PEC). On a PEC boundary Γ_{PEC} the tangential electric field vanishes: n×E=0. This condition enforces the correct boundary behavior at ideal conductors and is routinely used in scattering and resonator problems.

Perfect magnetic conductor (PMC). The dual condition n×H=0 enforces zero tangential magnetic field on an ideal magnetic wall (used

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primarily as a theoretical or symmetry boundary condition).

Local absorbing boundary conditions (ABCs). Classical absorbing BCs such as those derived by Engquist & Majda are local approximate conditions designed to reduce reflections from truncation boundaries for outgoing waves; they are simple to implement but have limited accuracy for oblique incidence and/or complicated geometries (Engquist, & Majda, 1977).

Perfectly matched layers (PMLs). The PML, introduced by Bérenger (1994) and developed in many follow-on formulations, is an artificial absorbing layer surrounding the computational domain that (in the continuous setting) is reflectionless for outgoing waves of any angle or frequency. In practice PMLs are implemented either via split-field formulations or via complex coordinate stretching; when discretized they are extremely effective at truncating open-region electromagnetic problems, though they require careful parameter selection and sometimes special treatment for anisotropic/inhomogeneous media (Bérenger, 1994; Johnson, 2021).

For FEM formulations the PEC (or impedance) boundaries enter naturally in the variational statement; ABCs or PMLs are added to model open-region radiation conditions and to produce well-posed truncated problems suitable for numerical solution (Johnson, 2021; Collino, & Monk, 1998).

Weak (Variational) Formulation Used in FEM

Finite element discretizations for Maxwell problems are typically derived from a variational (weak) formulation posed in the Hilbert space $H(\mathrm{curl};\Omega):=\{v\in(L^2(\Omega))^3:\nabla\times v\in(L^2(\Omega))^3\}$, which enforces the correct tangential continuity of the electric field across element faces. A standard frequency-domain variational problem (after multiplication by a complex test function $v\in H(\mathrm{curl};\Omega)$ and integration by parts) is:

Find $E\in H(\mathrm{curl};\Omega)$ satisfying the essential boundary conditions such that for all $v\in H(\mathrm{curl};\Omega)$

$$\int_\Omega \mu^{-1}(\nabla\times\mathbf{E})\cdot(\nabla\times\overline{\mathbf{v}})\,d\mathbf{x}-\omega^2\int_\Omega \varepsilon\,\mathbf{E}\cdot\overline{\mathbf{v}}\,d\mathbf{x}+\langle(\text{boundary/impedance terms})\rangle=i\omega\int_\Omega \mathbf{J}\cdot\overline{\mathbf{v}}\,d\mathbf{x},$$

where the overbar denotes complex conjugation for the usual sesquilinear form in the frequency domain. This formulation is the basis for conforming H(curl) finite elements (Nédélec edge elements) which ensure proper tangential continuity and avoid spurious eigenmodes that plague nodal discretizations. For transient (time-domain) FEM the weak form is analogous but includes time derivatives and is combined with suitable time integration schemes; discussions of time-domain FEM formulations, stability and dispersion are available in standard FEM monographs (Johnson, 2021; Hiptmair, 2002).

Practical FEM implementations may also use mixed formulations (introducing auxiliary potentials or Lagrange multipliers to enforce divergence constraints) or discontinuous Galerkin / hybridizable DG approaches that alter the structure of the discrete variational system; these alternatives trade off coupling pattern, sparsity, and implementational complexity and are active areas of research (Monk, 2003; Nguyen *et al.*, 2011).

Algebraic Systems and Their Properties

When the variational formulation above is discretized with H(curl) finite elements (e.g., low- or high-order Nédélec elements) the result is a complex-valued linear system of the form

$Ae=b$,

where the system matrix A encodes the curl–curl bilinear form and the negative mass ($-\omega^2\varepsilon$) term, together with any boundary or PML contributions. Several key algebraic properties follow and strongly influence solver choice:

Complex-valued entries. In the time-harmonic setting $\omega\neq0$ the mass term and boundary/PML terms normally introduce complex coefficients; hence both the matrix and right-hand side are complex, and solvers must handle complex arithmetic (Johnson, 2021).

Indefiniteness (and frequency dependence). The curl–curl operator combined with the $-\omega^2\varepsilon$ term produces an indefinite (non-coercive) operator for many frequencies of interest (the operator is coercive only under restrictive assumptions, e.g., imaginary part of permittivity or low frequencies). Indefiniteness is especially consequential at higher frequencies (the “wave” or Helmholtz-like regime), where iterative solvers slow down and preconditioning becomes more delicate. The resolution requirement (mesh size h relative to wavelength λ) produces the usual

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dispersion/pollution effects as frequency increases (Verfürth, 2017; Gopalakrishnan *et al.*, 2004).

Saddle-point structure for mixed formulations. Mixed or constrained formulations that explicitly enforce divergence-free conditions can produce block saddle-point systems; these need specialized block preconditioners or Schur-complement strategies. Several effective preconditioning frameworks (block preconditioners, augmentation methods, auxiliary-space decompositions) were developed to address this structure (Greif, & Schötzau, 2007).

Large scale / memory pressure. Realistic 3D problems, especially high-frequency or fine geometric detail lead, to very large numbers of degrees of freedom (DOFs). This causes heavy memory use and makes direct solvers impractical beyond moderate problem sizes; scalable iterative solvers with effective preconditioners and parallel implementations are therefore essential for production-scale simulations (Jin, 2015).

Together these properties explain why much of the numerical-analysis and HPC literature for Maxwell's equations is devoted to preconditioning (auxiliary-space and domain-decomposition approaches), multilevel solvers for H(curl) operators, and careful boundary truncation (PML/ABC) strategies. Foundational work and surveys that address these algebraic features and solver remedies include Hiptmair's survey on finite elements in computational electromagnetism, the Hiptmair–Xu auxiliary-space approach for H(curl) preconditioning, and a sequence of papers on preconditioners for mixed/time-harmonic Maxwell systems (Hiptmair, 2002; Hiptmair, & Xu, 2007; Greif, & Schötzau, 2007) .

Finite Element Discretization of Maxwell's Equations

This section summarizes standard and state-of-the-art finite-element discretizations used for Maxwell problems, focussing on (i) the choice of function spaces (Nédélec edge elements and their higher-order/hp extensions, and mixed formulations), (ii) the numerical mechanisms used to enforce/diverge the divergence constraint and suppress spurious modes, (iii) the trade-offs between accuracy and computational cost (including dispersion/pollution at high frequency and hp-strategies), and (iv) practical approaches for handling complex and multiscale geometries. All statements below are supported by the numerical-analysis and engineering literature cited at the ends of the principal paragraphs.

Choice of Function Spaces: Nédélec Edge Elements, Higher-Order Elements and Mixed Formulations

The preferred conforming finite element space for electromagnetic field unknowns is the curl-conforming Hilbert space H(curl;Ω). Nédélec's family of edge (vector) elements provides finite-dimensional subspaces of H(curl) with the correct tangential continuity across element faces; they were introduced precisely to avoid the breakdowns and non-physical ("spurious") eigenmodes observed when using standard nodal (Lagrange) elements for Maxwell problems. In three dimensions the two families of Nédélec elements (low/first family and higher-order / second family) are now standard building blocks of conforming FEM codes for Maxwell problems (Nédélec, 1980).

Higher-order (p) and hp finite element variants of Nédélec elements bring increased approximation power per degree of freedom and can reduce dispersion (phase) error for wave problems when used appropriately; hp-adaptive strategies have been developed specifically for Maxwell's equations and are implemented in several research codes and monographs. The theoretical underpinnings for stable high-order H(curl) discretizations are tied to projection-based interpolation and polynomial exact sequences that respect the discrete de Rham complex, an observation that underlies much of modern high-order FE design for electromagnetics (Demkowicz, 2006; Demkowicz, 2008).

Mixed formulations — for example introducing a scalar Lagrange multiplier to weakly enforce $\nabla\cdot(\varepsilon E)=\rho$ or pairing the electric and magnetic fields in a first-order formulation, are widely used when (i) divergence constraints must be treated explicitly (e.g., in low-frequency eddy-current problems or in coupled multi-physics settings), or (ii) when particular solver/preconditioner structures are desired. Mixed approaches typically produce block-structured (saddle point) algebraic systems and therefore require appropriate block preconditioning or Schur-complement strategies. The mixed-FEM literature (and comprehensive texts on mixed methods) gives both practical recipes and the stability theory required to design robust mixed discretizations for Maxwell problems (Boffi *et al.*, 2013).

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Treatment of Divergence-Free Constraints and Suppression of Spurious Modes

A persistent numerical challenge for finite element discretizations of Maxwell's equations is accurate enforcement of the divergence constraint and the elimination of spurious (non-physical) solutions, particularly in eigenvalue/resonance computations. Two complementary ideas have seen extensive use:

Use of compatible (conforming) spaces. By discretizing in the sequence of spaces that mirrors the continuous de Rham complex (e.g., nodal → edge → face → cell spaces), one automatically preserves the exactness properties that prevent many spurious phenomena; this is the compatibility viewpoint formalized by the finite element exterior calculus. Conforming Nédélec elements (and their higher-order analogues) are the canonical realization of this idea for H(curl) problems (Arnold, *et al.,* 2006).

Augmented/mixed or constrained formulations. When compatibility alone is insufficient (for example, on meshes with irregularity or when material discontinuities induce near-kernel fields), mixed formulations with explicit enforcement of divergence constraints (via Lagrange multipliers or penalty/augmentation terms) reduce null-space pollution and yield algebraic systems better suited to block preconditioners. The mixed-method theory and practical preconditioner architectures for these systems are discussed in depth in the mixed-FEM literature (Boffi, *et al.,* 2013).

Practically, the combined strategy, use of compatible H(curl) spaces plus careful treatment of boundary/constraint terms, is the accepted route to avoid spurious eigenmodes and non-physical solutions in production FEM codes for electromagnetics. Benchmarks and theoretical analyses in the literature demonstrate that violating compatibility (e.g., by using naive nodal discretizations) often produces spurious spectrum and severe accuracy degradation (Halla, & Oberender, 2025; Nédélec, 1980).

Accuracy Versus Computational Cost: Dispersion/Pollution, P/Hp Methods and Practical Choices

Wave propagation problems exhibit two linked sources of numerical error: the best-approximation error of the finite element space, and an additional *pollution* (dispersion) error that grows with the wave number if the discretization is not chosen carefully. Foundational analyses of the pollution effect for Helmholtz-type problems (closely related to frequency-domain Maxwell equations) show that naive h-refinement (reduce mesh size *h* only) becomes prohibitively expensive at high frequencies; by contrast, p- and hp-strategies can achieve higher accuracy per DOF and ameliorate pollution when projection-based interpolation and proper hp-refinement criteria are used. The literature therefore emphasizes hybrid strategies (hp-refinement, higher-order basis, and targeted adaptivity) for high-frequency electromagnetic simulations (Ihlenburg, & Babuska, 1997).

From an algorithmic cost perspective, increasing polynomial order (p-refinement) improves dispersion properties but causes element matrices to be denser and more expensive to assemble and apply; conversely, fine h-refinement increases DOF count and memory footprint and can negatively affect solver scalability. The optimal choice depends on the frequency regime, geometry complexity, available solver infrastructure (direct vs iterative solvers, preconditioners), and parallel architecture. Work on automated hp-adaptivity for Maxwell's equations demonstrates that reliable a-posteriori error indicators and adaptivity policies can guide these trade-offs in practice (Demkowicz, 2005).

Handling Complex and Multiscale Geometries

Realistic electromagnetic applications frequently involve complex CAD geometries, thin coatings, sharp edges, and localized features smaller than the wavelength all of which complicate meshing and discretization strategies. There are several practical approaches adopted in modern FEM workflows:

- Robust tetrahedral meshing and geometry preprocessing. Open and well-tested mesh generators (e.g., Gmsh for geometry/mesh pipelines, TetGen for quality tetrahedral meshes) are commonly used to convert CAD models into simulation-ready meshes with local refinement near features of interest. These tools support boundary conformity, sizing fields, and local refinement that are essential for resolving thin layers and fine geometric detail (Geuzaine & Remacle, 2009).
- Local mesh refinement and hp-adaptivity. Adaptive strategies that refine either h or p locally, guided by a-posteriori indicators or goal-oriented error estimates, can concentrate DOFs where they are needed and keep global problem size manageable. In three-dimensional Maxwell problems, hp-adaptive schemes (with projection-based operators that respect the discrete de Rham sequence) have

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been demonstrated to be effective for singularities induced by corners, material jumps, and sharp features (Demkowicz, 2006).

- Specialized discretizations for multiscale features. Where sub-wavelength features are present (thin dielectric coatings, metasurface elements), approaches include targeted local refinement, homogenized or multiscale models for subgrid features, or hybrid discretizations (e.g., combining surface integral equation representations for thin conductors with volume FEM for surrounding media). The choice among these depends on required fidelity, available solver capacity, and the degree to which small features can be approximated analytically or homogenized. Representative literature on multiscale and hybrid modeling for electromagnetics presents practical guidelines and benchmarks (see cited review material, Hiptmair, 2002).

Practical Implications for Solver Design and Software

The discretization choices above have direct consequences for linear algebra and solver design:

- H(curl) conforming Nédélec spaces produce system matrices with sparsity patterns and null-space structure that drive the design of preconditioners (auxiliary-space methods, multigrid on curl-conforming spaces, domain decomposition tailored to edge elements (Hiptmair, 2002).
- High-order and hp discretizations increase per-element computational cost and change the relative performance of direct versus iterative solvers; they typically benefit from matrix-free operator evaluation and optimized local element kernels in HPC settings (Demkowicz, 2006).
- Meshing and adaptivity workflows must be integrated with solver infrastructure to permit dynamic load balancing and scalable assembly on parallel architectures a recurrent theme in high-performance FEM libraries and exascale readiness efforts (Geuzaine, & Remacle, 2009).

COMPUTATIONAL CHALLENGES IN SCALABLE FEM FOR MAXWELL'S EQUATIONS

The finite element discretization of Maxwell's equations produces large, sparse, and often indefinite linear systems whose efficient solution is one of the main bottlenecks in large-scale computational electromagnetics (CEM). These computational challenges arise from both the mathematical properties of Maxwell operators and the practical constraints of modern high-performance computing (HPC).

System Size and Memory Demands

Large-scale three-dimensional simulations, such as antenna arrays, photonic crystals, and biomedical imaging, involve discretizations with millions to billions of unknowns. The number of degrees of freedom (DOFs) grows rapidly with frequency due to the need to resolve multiple wavelengths per domain, especially when high-order or hp-FEM is employed (Jin, 2015; Monk, 2003). As a result, the memory required to store sparse system matrices and intermediate solver states can exceed the capacity of a single compute node, necessitating distributed-memory parallelization (Demmel, *et al.*, 2012; Andrej, *et al.*, 2024).

Indefiniteness and Conditioning of Linear Systems

Maxwell systems, particularly in frequency-domain formulations, are indefinite due to the oscillatory nature of wave equations. This indefiniteness significantly reduces the robustness of standard iterative solvers like conjugate gradient (CG), which assume positive definiteness (Greif & Schötzau, 2007). Additionally, the discretized curl-curl operator leads to ill-conditioned systems at low frequencies (the so-called *"low-frequency breakdown"*) and near resonances (Hiptmair, 2002; Hiptmair & Xu, 2007). These conditioning issues complicate both direct and iterative solution strategies, increasing the need for specialized preconditioning techniques.

Solver Scalability on Modern Architectures

Achieving scalability on modern HPC systems requires algorithms that minimize communication and balance computation across thousands of processors; this is especially important for sparse linear algebra arising from large 3D FEM discretizations. Classical direct solvers (multifrontal/supernodal methods) are robust but often scale poorly in time and memory for very large 3D problems because of factorization fill-in and global communication costs. Multifrontal approaches have been parallelized effectively (e.g., asynchronous multifrontal implementations), but their superlinear memory/time behavior still limits applicability to extreme-scale problems (Amestoy, *et al.*, 2001).

Preconditioned iterative solvers (Krylov methods) combined with multigrid or domain-decomposition

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preconditioners are generally more scalable on distributed architectures, but their efficiency depends strongly on mesh quality, domain partitioning, and the construction of effective coarse spaces and transmission conditions; the design of coarse spaces and constraint selection is central to ensuring robust, mesh-independent convergence in domain-decomposition frameworks such as BDDC/FETI-DP (Dohrmann, 2003; Hiptmair, & Xu, 2007; Toselli, & Widlund, 2004).

GPU-accelerated and other heterogeneous architectures offer large amounts of fine-grained parallelism and memory bandwidth, but exploiting them requires careful reorganization of assembly, preconditioning, and solver kernels to avoid memory-bound bottlenecks, preserve numerical stability (precision / arithmetic concerns), and limit host–device communication. Microbenchmarking and architecture studies illustrate that memory hierarchy and data movement are critical considerations when porting FEM solvers to GPUs (Mei, & Chu, 2016).

Treatment of Complex and Multiscale Geometries

Simulating realistic electromagnetic devices often involves domains with highly heterogeneous materials and multiscale features, from subwavelength resonators to large enclosing domains. Such problems demand adaptive mesh refinement (AMR) and high-order or spectral/hp curved elements to accurately represent curved boundaries and fine features without incurring prohibitive numbers of degrees of freedom (DOFs). For example, *hp-Adaptive Finite Elements for Maxwell's Equations* (Demkowicz, 2003) describes automated hp-refinement techniques for Maxwell problems to balance accuracy and DOF growth. Similarly, multiscale hp-FEM for photonic crystal bands demonstrates that multiscale basis functions distinguish microstructure while using a coarser macroscopic mesh to maintain computational efficiency (Brandsmeier, *et al.,* 2011). However, AMR introduces challenges in parallel implementations: unbalanced meshes lead to load-imbalance, complex mesh partitioning, and increased communication overhead (e.g., in parallel hp codes such as par3Dhp) (Paszyński, & Demkowicz, 2006). Meanwhile, high-order elements (spectral or hp) improve approximation but raise the cost of matrix assembly, increase condition numbers, and demand more sophisticated preconditioners and quadrature rules. The trade-offs between higher accuracy and computational intensity are illustrated, for instance, in the reaction-diffusion surface problem using spectral/hp discretisation by Cantwell *et al.*, 2014 (Cantwell, *et al.*, 2014).

Absorbing Boundary Conditions and PML Implementation

The accurate truncation of unbounded domains with absorbing boundary conditions, especially perfectly matched layers (PML), introduces further computational challenges. PML formulations lead to complex-valued, anisotropic coefficients that exacerbate conditioning issues in linear systems (Chew & Weedon, 1998; Bérenger, 1994). Their correct implementation in scalable solvers requires careful handling of anisotropy in both discretization and preconditioning strategies (Collino, & Monk, 1998).

SCALABLE SOLVER STRATEGIES FOR FEM IN MAXWELL'S EQUATIONS

The computational challenges outlined in Section 4 necessitate solver technologies that can handle the large, indefinite, and ill-conditioned systems resulting from finite element discretizations of Maxwell's equations. Scalable solver strategies must combine mathematical robustness with efficient parallelization across modern high-performance computing (HPC) architectures. This section reviews key approaches, focusing on multigrid methods, domain decomposition, auxiliary-space preconditioners, and emerging GPU/exascale implementations.

Direct vs. Iterative Solvers

Direct solvers, such as multifrontal methods (Amestoy *et al.*, 2001), offer robustness but scale poorly due to memory and communication bottlenecks, limiting their use to moderate problem sizes. Iterative solvers, particularly Krylov subspace methods like GMRES, BiCGStab, and MINRES, are more memory-efficient and parallelizable but require effective preconditioners to converge reliably for indefinite Maxwell systems (Greif, & Schötzau, 2007). Consequently, the focus in large-scale CEM has shifted toward iterative-preconditioned methods.

Multigrid Methods

Multigrid is among the most powerful scalable strategies for PDE systems, but its application to Maxwell's equations is nontrivial. Classical geometric multigrid (GMG) suffers from difficulties in constructing appropriate coarse spaces for curl-conforming fields. Algebraic

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multigrid (AMG) variants tailored to H(curl) spaces, such as those implemented in BoomerAMG and HYPRE, address this limitation by exploiting the vector field structure (Arnold *et al.,* 2000). Smoothed aggregation AMG has shown particular promise for Maxwell systems (VanÄek *et al.,* 1996). Hybrid approaches, where multigrid is combined with auxiliary-space preconditioners, further improve convergence rates (Hiptmair & Xu, 2007).

Domain Decomposition Methods

Domain decomposition (DD) methods divide the computational domain into subdomains solved locally with inter-subdomain coupling. Non-overlapping approaches such as FETI-DP (Finite Element Tearing and Interconnecting – Dual–Primal) and BDDC (Balancing Domain Decomposition by Constraints) have demonstrated strong scalability for Maxwell problems (Dohrmann, 2003; Li & Widlund, 2006). Optimized Schwarz methods with impedance transmission conditions further accelerate convergence in wave-dominated regimes (Gander, 2006). DD methods are particularly well suited to distributed-memory HPC systems, where local solves map naturally to parallel processes.

Auxiliary-Space Preconditioning

The auxiliary-space preconditioning (ASP) framework, introduced by Hiptmair and Xu (2007), is a breakthrough for Maxwell systems. ASP maps difficult H(curl)- or H(div)-problems to better-understood H^1 (scalar Laplacian) problems, for which scalable solvers are well established. This reduces the effective condition number and accelerates Krylov solvers. Extensions of ASP to high-order and adaptive discretizations further broaden its applicability in CEM (Arnold *et al.,* 2010).

Hybrid and Block Preconditioners

Block-diagonal and block-triangular preconditioners exploit the structure of mixed formulations of Maxwell's equations, separating curl–curl and divergence constraints (Greif & Schötzau, 2007). Recent advances combine block methods with multigrid and auxiliary space preconditioners for improved robustness across frequency ranges.

Hybrid preconditioners that integrate coarse-space corrections from domain decomposition with fine-level multigrid smoothers are emerging as effective strategies. For example, Bonazzoli et al. (2019) develop a two-level overlapping Schwarz preconditioner with coarse space built using low-order elements, showing wavenumber-independent GMRES convergence in absorptive Maxwell problems (Bonazzoli *et al.,* 2019). Similarly, Chen, *et al.,* (2018) propose block diagonal and block triangular preconditioners using multigrid V-cycles for mixed finite element systems of the vector Laplacian and Maxwell equations with divergence constraints (Chen *et al.,* 2018).

HPC and GPU-Accelerated Implementations

With the advent of GPU-accelerated clusters and exascale computing, solver design must minimize communication and exploit fine-grained parallelism. Libraries such as MFEM (Anderson *et al.,* 2021), and frameworks like PETSc and Trilinos, have developed GPU-enabled FEM and solver kernels using CUDA, HIP, and SYCL backends. Mixed-precision methods, including iterative refinement in three precisions (Carson & Higham, 2018), and communication-avoiding Krylov solvers such as CA-GMRES (Hoemmen, 2010) further enhance scalability on modern architectures. Exascale projects such as CEED (Center for Efficient Exascale Discretizations) demonstrate that careful kernel optimization and algorithmic reformulation are essential for achieving performance at scale (Pazner *et al.,* 2023).

APPLICATIONS AND CASE STUDIES IN SCALABLE FEM FOR MAXWELL'S EQUATIONS

Scalable finite element methods (FEM) for Maxwell's equations have enabled progress in diverse areas of science and engineering where electromagnetic phenomena must be accurately modeled in complex geometries. This section highlights representative applications, ranging from antenna design and photonics to metamaterials and biomedical imaging, that demonstrate both the challenges and opportunities for large-scale FEM simulations.

Antenna and Radar System Design

Electromagnetic modeling of antenna arrays and radar systems requires the simulation of open-domain scattering problems with highly heterogeneous geometries. FEM combined with perfectly matched layers (PML) provides accurate truncation for open domains, enabling the analysis of antenna radiation patterns, mutual coupling, and radar cross-sections (Jin, 2015). Scalable FEM solvers are particularly valuable for phased-array systems and electrically large structures, where millions of degrees of freedom (DOFs) are

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necessary to resolve fine geometric features and high-frequency effects (Cessenat, 1996; Collino, & Monk, 1998). High-order elements and domain-decomposition preconditioners have been shown to reduce computational time in practical antenna design workflows (Graglia, 2002).

Photonics and Optical Waveguides

In integrated photonics, FEM is widely applied to simulate waveguides, resonators, and photonic crystals with subwavelength features. Accurate modeling often uses higher-order vectorial elements and adaptive mesh refinement to resolve sharp material interfaces and capture Bloch eigenmodes (Burger *et al.,* 2005). As device designs move toward three-dimensional photonic crystal slabs and densely coupled resonator arrays, the problem sizes increase dramatically, making scalability in FEM performance critical. Hybrid solver approaches, for example, combining FEM with multigrid techniques and auxiliary-space preconditioners, have shown promise in other computational electromagnetics settings and are attractive for photonic device analysis. Foundational texts such as *Photonic Crystals: Molding the Flow of Light* (Meade, *et al.,* 2008) provide theoretical and computational background for such photonic crystals and waveguide structures.

Metamaterials and Electromagnetic Cloaking

Metamaterials, including negative-index media and cloaking devices, present multiscale modeling challenges due to subwavelength resonant structures embedded in large host domains. FEM's flexibility in handling inhomogeneous and anisotropic media makes it a method of choice for such applications (Pendry *et al.,* 2006). However, the strong dispersion and anisotropy of metamaterials yield highly ill-conditioned systems, necessitating robust scalable solvers (Hiptmair & Xu, 2007). Case studies of cloaking simulations demonstrate that scalable FEM enables the exploration of full 3D cloaks, which would otherwise be computationally infeasible (Kwon & Werner, 2008).

Biomedical Imaging and Therapy

In biomedical engineering, FEM-based Maxwell and electromagnetic solvers are used in hyperthermia treatment planning and MRI coil design. These applications require accurate modeling of human tissues with highly heterogeneous electromagnetic properties (Poljak & Cvetkovic, 2019) and often involve anatomically detailed models derived from MRI or CT scans. For example, Li et al. (2011) used a 3D FEM electromagnetic solver together with SAR/thermal simulations on a detailed anatomical body model for planning hyperthermia treatments (Li *et al.,* 2011). Further, computational modeling and real-time control have been demonstrated for laser ablation therapies using bio-heat transfer FEM frameworks; these setups involve feedback control based on imaging and rapid solution of the thermal conduction/EM coupling (Fuentes *et al.,* 2009). Simulations with large vascular networks and realistic anatomies have used GPU-accelerated solvers to speed up thermal modeling (Kok *et al.,* 2013), though achieving fully real-time Maxwell FEM in imaging/treatment planning remains an active research goal.

Multiphysics and Coupled Problems

Many practical applications require coupling Maxwell's equations with other physical models, such as heat transfer and structural mechanics in high-power devices or electromagnetic–thermal interactions in biomedical therapies. FEM is particularly well suited for such multiphysics simulations because its variational framework naturally accommodates multiple PDEs within a single discretization scheme. However, scalability challenges intensify as different physical subsystems interact, often leading to ill-conditioned block structures and nonlinear feedback between fields.

Recent advances in multiphysics simulation frameworks, such as the MOOSE platform (Permann *et al.,* 2020), demonstrate how modular FEM infrastructures can scale to thousands of processors while supporting coupled physics including electromagnetics, thermal conduction, and mechanics. Research on block preconditioning for coupled PDE systems provides further insight: for example, Elman *et al.,* (2002) analyzed block preconditioners for the incompressible Navier–Stokes equations, establishing mesh-independent convergence and robustness (Elman, *et al.,* 2002). More recently, Egermaier & Horníková (2024), investigated approximate block preconditioners in the context of isogeometric discretizations (Egermaier, & Horníková, 2024). These developments suggest that scalable FEM algorithms are increasingly capable of handling the multiphysics demands of modern engineering applications, from high-power electronics to coupled electromagnetic–thermal–mechanical systems.

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FUTURE DIRECTIONS AND OPPORTUNITIES

The development of scalable finite element methods (FEM) for solving Maxwell's equations remains an active frontier in computational mathematics and engineering. As electromagnetic applications demand higher accuracy, larger problem sizes, and more complex geometries, progress hinges on advances in algorithms, hardware utilization, and interdisciplinary integration. Several promising directions are outlined below.

High-Performance Computing (HPC) and Exascale Readiness

The rapid growth of parallel computing architectures, including GPUs and heterogeneous clusters, presents opportunities to push FEM-based solvers to unprecedented scales. Exascale computing initiatives will enable the simulation of multi-billion degree-of-freedom electromagnetic systems, but achieving scalability requires domain decomposition methods, communication-avoiding algorithms, and efficient preconditioners tailored to Maxwell's equations (Kolev, *et al.,* 2021; Bollhöfer, *et al.*, 2009).

Advanced Preconditioning and Iterative Solvers

The indefinite and complex-valued nature of the algebraic systems from FEM discretizations necessitates more robust iterative methods. Future work will likely focus on multigrid techniques, hybrid direct-iterative solvers, and physics-informed preconditioners that preserve divergence constraints while reducing computational cost (Hiptmair, & Xu, 2007; Phillips, *et al.*, 2018).

Integration of Multiscale and Multiphysics Problems

Modern applications often involve coupled physical phenomena for example, electromagnetics with thermal, mechanical, or fluidic effects in biomedical imaging or antenna design. Scalable FEM must extend to multiphysics formulations without compromising stability or efficiency, opening opportunities for unified solvers across domains (Zienkiewicz, *et al.*, 2013; Monk, 2003).

Geometric Complexity and Adaptive Methods

As device geometries grow more intricate, ranging from metamaterials to photonic crystals, adaptive mesh refinement (AMR) and higher-order or hp finite elements become increasingly important for resolving fine geometric features without excessive computational cost. Combining AMR/hp-adaptive methods with efficient solver technologies offers opportunities for localized accuracy while maintaining global efficiency. Reliable posteriori error estimation for Maxwell's equations (Schöberl, 2008) and comprehensive hp-adaptive frameworks in three dimensions (Kurtz, *et al.*, 2008) provide solid theoretical and algorithmic foundations for these directions.

Machine Learning for FEM Acceleration

Emerging research is illustrating the potential of machine learning (ML) to accelerate FEM solvers by providing surrogate models, guiding adaptive meshing strategies, and reducing time-to-solution for parametric studies. For example, the work *"Multi-fidelity surrogate modeling through hybrid machine learning for biomechanical and finite element analysis of soft tissues"* (2022) demonstrates how low-fidelity FE results combined with ML can approximate expensive FE solutions in soft tissue mechanics efficiently (Sajjadinia, *et al.*, 2022). Similarly, *G-Adaptivity: Optimised graph-based mesh relocation for finite element methods* (Rowbottom, *et al.*, 2024) uses a graph neural network to relocate mesh points and reduce FE error while keeping computational cost manageable. These developments suggest that integrating ML surrogates and adaptive meshing decision models into FEM workflows promises significant speedups, particularly in design optimization and real-time or near-real-time settings.

Open-Source Frameworks and Community Development

Sustained progress in scalable FEM will also depend on robust open-source platforms that integrate advanced solvers, HPC capabilities, and domain-specific tools. Frameworks such as MFEM (Anderson, *et al.*, 2021), FEniCS (Logg *et al.*, 2012), and PETSc (Balay, *et al.*, 2020) provide solid foundations, but broader collaboration between applied mathematicians, computer scientists, and engineers will be essential to accelerate innovation and ensure these tools adapt to emerging HPC architectures.

CONCLUSION

Finite element methods (FEM) are essential for solving Maxwell's equations in complex geometries and have enabled progress in electromagnetics, photonics, antenna design, and biomedical applications. However, important challenges remain. The algebraic systems from FEM are indefinite and large in scale, enforcing divergence-free constraints is difficult, and intricate geometries are costly to represent

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accurately. These factors continue to limit scalability.

This paper has outlined the mathematical foundations of Maxwell's equations, the role of weak formulations, and the use of edge-based finite elements for stable discretizations. It has also identified major computational bottlenecks, including spurious modes, solver inefficiencies, and the balance between accuracy and cost in multiscale problems. Together, these issues emphasize the need for scalable FEM solvers.

Looking forward, progress will depend on advances in high-performance computing (HPC) and exascale systems, along with algorithms and preconditioners adapted for large parallel environments. Multiphysics and multiscale formulations provide new opportunities but add further complexity. Adaptive methods, higher-order discretizations, and machine learning acceleration offer promising paths to improved efficiency. Finally, open-source frameworks and community-driven platforms will play a key role in sustaining innovation.

In summary, scalable FEM is not only a numerical goal but a foundation for addressing pressing scientific and engineering challenges.

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