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An Integral Representation Formula for First-Order Elliptic Systems with Constant Coefficients in Layer-Type Domains

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Abstract: This paper investigates the validity of an integral representation formula for a vector function satisfying a first-order elliptic system with constant coefficients in an unbounded domain of layer type. The study builds upon earlier results concerning bounded domains, extending the applicability of such formulas to broader unbounded settings. Drawing from foundational works by Tarkhanov, Yarmukhamedov, and Bitsadze, the authors present a rigorous framework to validate the integral formula under growth conditions imposed on the solution and the domain boundary. The proof is structured through the introduction of specially constructed entire functions and asymptotic estimates, which guarantee the convergence of the integral representation. Two theorems are proved: the first establishes the formula's validity for vector functions with controlled growth inside the domain; the second generalizes the result to more inclusive function classes with boundary growth conditions. The results contribute to the theory of elliptic systems by providing tools for solving boundary value problems in more complex, unbounded geometries.
Keywords: Systems of equations, elliptic system, integral formula, unbounded domain, fundamental solution.

INTRODUCTION

This paper considers the validity of an integral representation formula in an unbounded domain for a growing vector function satisfying a first-order elliptic system with constant coefficients. The integral formula serves as a fundamental tool for solving various boundary value problems for systems of equations.

In a bounded domain, an integral representation formula for a vector function satisfying a first-order elliptic system with constant coefficients was proved by (Tarkhanov, N. N. 1980).

For the Laplace equation, an integral formula in an unbounded domain within the class of growing functions was established by Sh. Yarmukhamedov (Yarmukhamedov, S. H. 1981).

Many mathematical problems are solved using integral formulas both in bounded and unbounded domains.

A spatial analogue of the Cauchy-type integral was studied by A.V. Bitsadze (Bisadze, A.V. 1953). A similar integral formula for the three-dimensional Cauchy–Riemann system in a bounded domain was presented in the book by A.V. Bitsadze (BisadzeM, A.V. 1984). In four-dimensional space, an integral formula for the Cauchy–Riemann system was given in the work of V.S. Vinogradov (Vinogradov, V. S. 1964).

Let the following designations be used:

$$x=(x_1,x_2,...,x_n),\ y=(y_1,y_2,...y_n),\ r=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+...+(y_n-x_n)^2},$$

$$x^1=(x_1,x_2,...,x_{n-1}),\ y^1=(y_1,y_2,...,y_{n-1}),\ \alpha=\sqrt{(y_1-x_1)^2+(y_2-x_2)^2+...,(y_{n-1}-x_{n-1})},$$

$$\omega=i\sqrt{u^2+\alpha^2}+y_n$$

$$u(x)=(u_1(x),u_2(x),...,u_m(x))^T,\ m\geq n,\ \frac{\partial}{\partial x}=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},...,\frac{\partial}{\partial x_n}\right)^T$$

$E(x)$ - is a diagonal matrix of size $m\times m$, $u^0=(1,1,...,1)$ m – is dimensional vector

Let $A_{l\times m}(x^T), l\geq n, m\geq n$ denote the class of matrices $D(x^T)\in A_{l\times n}(x^T)$ whose elements are linear functions with complex coefficients satisfying the condition, where

$D^*(x^T)$ *is the Hermitian adijoint of the matrix* $D(x^T)$.

Let $G\subset R^n$ be a simply connection bounded domain with a smooth boundary.

In the domain G, a system of differential equations of the form

$$D\left(\frac{\partial}{\partial x}\right)u(x)=0,\ x\in G, \qquad (1),$$

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where $D(x) \in A_{l\times n}(x^T)$.

If $u(x) = C^1(G) \cap C(\overline{G})$ satisfies system (1) in the domain G,then the following integral representation formulas holds:

$$u(x) = \int\limits_{\partial G} M(x,y) u(y) ds_y, x \in G, \quad (2),$$

where $M(x,y) = \left(E\left(\frac{C_m}{r^{m-2}} \right) D^* \left(\frac{\partial}{\partial x} \right) \right) D(t^T),$ (3).

Problem Statement. This work establishes the validity of the integral formula (2) for growing solutions of elliptic systems with constant coefficients in an unbounded domain of layer type.

To prove the integral formula, the following function is introduced in (Yarmukhamedov, S. H. *et al.,* 1992).

Let $K(w), w = u + iv$ ($u, v -$real-valued) be an entire function, real-valued rof real w, satisfying the condition

$$K(u) \neq 0, \sup_{v \geq 1} \left| v^p K^p(w) \right| = M(p,u) < \infty, -\infty < u < \infty,$$

$$s = \alpha^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + ... + (y_{m-1} - x_{m-1})^2, r = |y - x|, \quad (4),$$

$$x = (x_1, x_2, ..., x_m), y = (y_1, y_2, ..., y_m) \in R^m, m \geq 2,$$

$$\sup_{v \geq 1} \left| vK^1(w) \right| = M_1(u) < \infty, m = 2$$

The function $\Phi(y,x)$ for $\alpha > 0$ is defined as follows.

If $m = 2,$ then

$$2\pi K(x_2) \Phi(y,x) = \int\limits_0^\infty \mathrm{Im} \frac{K\left(i\sqrt{u^2 + \alpha^2} + y_2\right)}{i\sqrt{u^2 + \alpha^2} + y_2 - x_2} \frac{udu}{\sqrt{u^2 + \alpha^2}}, \quad (5)$$

If $m = 2n + 1.n \geq 1,$ then

$$C_m K(x_m) \Phi(y,x) = \frac{\partial^{n-1}}{\partial s^{nn-1}} \int\limits_0^\infty \mathrm{Im} \frac{K\left(i\sqrt{u^2 + \alpha^2} + y_m\right)}{i\sqrt{u^2 + \alpha^2} + y_m - x_m} \frac{du}{\sqrt{u^2 + \alpha^2}} \quad ,(6)$$

Where

$$C_m = (-1)^{n-1} 2^{-n} (m-2) \pi \omega_m (2n-1)!\,,$$

If $m = 2n, n \geq 2,$ then

$$C_m K(x_m) \Phi(y,x) = \frac{\partial^{n-1}}{\partial s^{n-1}} \mathrm{Im} \frac{K(i\alpha + y_m)}{\alpha(i\alpha + y_m - x_m)} \quad ,(7)$$

Where,

$C_m = (-1)^{n-1}(n-1)!(m-2)\omega_m$, $\omega_m -$ is the surface area of the unit sphere in R^m.

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In paper (Yarmukhamedov, S. H. *et al.*, 1992)., it is shown that the function $\Phi(y,x)$ can be represented in the form

$$\Phi(y,x)=\frac{1}{2\pi}\ln\frac{1}{r}+\psi(y,x),m=2,$$
$$\Phi(y,x)=\frac{r^{2-m}}{\omega_m(m-2)}+\psi(y,x),m\geq 3, \quad (8)$$

where $\psi(y,x)-$ is a harmonic function everywhere in $R^m,m\geq 2$ is variables y ,including the point $y=x$.

Let the domain G be an unbounded domain with a piecewise smooth boundary. Let K_R denote the ball of radius R centered at the origin, and let $G_R=G\cap K_R, G_R{}^{\infty}=G\setminus K_R$. The integral formula is valid in the unbounded domain G if the following condition is satisfied:

$$\lim_{R\to\infty}\int_{\partial G_R{}^{\infty}} M(x,y)u(y)ds_y=0, x\in G, \quad (9)$$

Indeed, in the domain G, the integral formula for the solution of system (1) is applied.

$$u(x)=\int_{\partial G} M(x,y)u(y)ds_y=\int_{\partial G_R} M(x,y)u(y)ds_y+\int_{\partial G_R{}^{\infty}} M(x,y)u(y)ds_y, x\in G \quad (10)$$

Considering equality (9) and taking the limit in equality (10), the integral formula in the unbounded domain follows.

Consider the case when the domain $G_\rho,\rho>0,$ lies within a layer of minimal width h, $0<y_3<h, h=\frac{\pi}{\rho},\rho>o$,and the boundary of the domain ∂G extends to infinity. Assume that for some $b_0>0$ the surface area of the boundary satisfies the growth condition.

$$\int_{\partial G}\exp\left\{-b_0 ch\rho_0\left|y'\right|\right\}ds<\infty, 0<\rho_0<\rho, \quad (11)$$

Theorem 1. Let $u(x)=C^1(G)\cap C(\overline{G})$ satisfy system (1) in the domain G , and also satisfy the growth condition.

$$\left|u(y)\right|\leq C\exp\left[\exp\rho_2\left|y'\right|\right],\rho_2<\rho, y\in G,$$
$$\left|y'\right|^2=y^2{}_1+y^2{}_2+...+y^2{}_{m-1} \quad (12)$$

Then the integral formula (2) holds in the unbounded domain G .

Proof. Let the function $K(w), w=u+iv$ be an entire function, when $\sigma>0$ in (5), (6), (7) we choose it as follows.

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$$K(w)=\frac{\exp\left[\sigma w-bchi\rho_1\left(w-\frac{h}{2}\right)-b_1chi\rho_0\left(w-\frac{h}{2}\right)\right]}{\left(w-x_m+3h\right)^k},$$

$$w=y_m+i\sqrt{u^2+\alpha^2},0<\rho_1<\rho,0<x_m<h,b>0, \qquad \text{(A)}$$

$$b_1>b_0\left(\cos\rho_0\frac{h}{2}\right)^{-1},$$

$\rho-$ a non-negative integer, $k-$ a natural number.

Then we denote the function $\Phi(y,x)$ as $\Phi(y,x)=\Phi_\sigma(y,x)$

When $m=2,$ we have.

$$2\pi K(x_2)\Phi(y,x)=\int_0^\infty \operatorname{Im}\frac{\exp\left[\sigma w-bchi\rho_1\left(w-\frac{h}{2}\right)-b_1chi\rho_0\left(w-\frac{h}{2}\right)\right]}{(i\sqrt{u^2+\alpha^2}+y_2-x_2)\left(w-x_2+3h\right)}\frac{udu}{\sqrt{u^2+\alpha^2}} \qquad (13)$$

$$K(x_2)=\frac{\exp\left[\sigma x_2-b\cos\rho_1\left(x_2-\frac{h}{2}\right)-b_1\cos\rho_0\left(x_2-\frac{h}{2}\right)\right]}{3h}$$

When $m=2n+1,n\geq 1,$

$$C_mK(x_m)\Phi(y,x)=\frac{\partial^{n-1}}{\partial s^{nn-1}}\int_0^\infty \operatorname{Im}\frac{\exp\left[\sigma w-bchi\rho_1\left(w-\frac{h}{2}\right)-b_1chi\rho_0\left(w-\frac{h}{2}\right)\right]}{(i\sqrt{u^2+\alpha^2}+y_m-x_m)\left(w-x_m+3h\right)^k}\frac{du}{\sqrt{u^2+\alpha^2}}$$

$$K(x_m)=\frac{\exp\left[\sigma x_m-bchi\rho_1\left(x_m-\frac{h}{2}\right)-b_1chi\rho_0\left(x_m-\frac{h}{2}\right)\right]}{(3h)^k},s=i\sqrt{u^2+\alpha^2}+y_m,(14)$$

When $m=2n,n\geq 2$ we have

$$C_mK(x_m)\Phi(y,x)=\frac{\partial^{n-1}}{\partial s^{n-1}}\operatorname{Im}\frac{\exp\left[\sigma(i\alpha+y_m)-bchi\rho_1\left(i\alpha+y_m-\frac{h}{2}\right)-b_1chi\rho_0\left(i\alpha+y_m-\frac{h}{2}\right)\right]}{\alpha\left(i\alpha+y_m-x_m\right)\left(i\alpha+y_m-x_m+3h\right)^k}$$

$$K(x_m)=\frac{\exp\left[\sigma x_m-b\cos\rho_1\left(x_m-\frac{h}{2}\right)-b_1\cos\rho_0\left(x_m-\frac{h}{2}\right)\right]}{(3h)^k},s=\alpha^2,(15).$$

To prove the theorem, we will show that condition (9) is satisfied.
For this, we estimate the integral representation of the solution to system (1) in the domain G_R .
The integral formula is composed of a superposition

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$\frac{\partial \Phi_\sigma(y,x)}{\partial y_i} u_j(y), i=\overline{1,m}, j=\overline{1,m}$ expression.

To estimate the above expression, we use the inequality.

$$
\begin{aligned}
&-\frac{\pi}{2} < -\frac{\rho_0 h}{\rho} < \frac{\rho_0}{\rho}\left(y_m - \frac{h}{2}\right) < \frac{\rho_0 h}{\rho} < \frac{\pi}{2}, 0 < \rho_0 < \rho, \\
&-\frac{\pi}{2} < -\frac{\rho_1 h}{\rho} < \frac{\rho_1}{\rho}\left(y_m - \frac{h}{2}\right) < \frac{\rho_1 h}{\rho} < \frac{\pi}{2}, 0 < \rho_1 < \rho,
\end{aligned} \tag{16}
$$

Consequently, we have the following inequalities.

$$
\begin{aligned}
&\cos \rho_0 \left(y_m - \frac{h}{2}\right) \geq \cos \rho_0 \frac{h}{2} \geq \delta_0 > 0, \\
&\cos \rho_1 \left(y_m - \frac{h}{2}\right) \geq \cos \rho_1 \frac{h}{2} \geq \delta_0 > 0, 0 < \rho_0, \rho_1 < \rho
\end{aligned} \tag{17}
$$

When $x \in G, y \to \infty, y \in G \cap \partial G$ is fixed, we obtain asymptotic estimates for the function $\Phi_\sigma(y,x)$ and its derivatives.

$$
\left|\Phi_\sigma(y,x)\right| + \left(\frac{\partial \Phi_\sigma(y,x)}{\partial x_i}\right) = O\left(\exp\left[-\varepsilon_1 ch\rho_1 \left|y'\right| - b_0 ch\rho_0 \left|y'\right|\right]\right), \varepsilon_1 > 0 \,. \tag{18}
$$

We choose ρ_1 with the following inequality $\rho_2 < \rho_1 < \rho$.

Using estimates (18), we obtain the fulfillment of condition (9); therefore, the integral formula (2) holds in the unbounded domain.

In the next theorem, we will prove the integral formula in an unbounded domain for growing vector functions that satisfy system (1) in more general classes of growing functions.

Theorem 2. Let $u(y)$ be a vector function from the class $u(y) \in C^1(G) \cap C(\overline{G})$ that satisfies system (1) in the domain G and has the following growth inside the domain, $\left|u(y)\right| \leq \exp\left[o\left(\exp \rho \left|y'\right|\right)\right], \left|y'\right|^2 = y_1^2 + y_2^2 + \dots y_{m-1}^2, y \to \infty, y \in G$,

as well as on the boundary of the domain has growth of the form,

$\left|u(y)\right| \leq C \exp\left[a \cos \rho_1 \left(y_m - \frac{h}{2}\right) \exp \rho_1 \left|y'\right|\right], \left|y'\right|^2 = y_1^2 + y_2^2 + \dots y_{m-1}^2, y \in \partial G, a \geq 0,$ Then the following integral formula is valid.

$$
u(x) = \int_{\partial G} M_\sigma(x,y) u(y) ds_y, x \in G, \tag{19}
$$

where
$$
M_\sigma(x,y) = \left(E\left(\Phi_\sigma(y,x)\right) D^*\left(\frac{\partial}{\partial x}\right)\right) D\left(t^T\right), \tag{20}
$$

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Proof. Using the plane $y_m = \frac{h}{2}$ we divide the domain G into two domains. $G_1 = \left\{(y); 0 < y_m < \frac{h}{2}\right\} u G_2 = \left\{(y); \frac{h}{2} < y_m < h\right\}$.For domain G_1, we define the function $K(w)$ as follows:

$$K_1(w) = K(w)\exp\left[-\delta chi\tau\left(w - \frac{h}{4}\right) - \delta_1 chi\left(w - \frac{h}{2}\right)\right]$$

(21),

$$w = y_m + iv, \delta_1 > 0, \delta > 0, \rho < \tau < 2\rho, 0 \le y_m \le \frac{h}{2}, 0 \le x_m \le \frac{h}{2}$$

Where function $K(w)$ is defined by formulas (13), (14), and (15). Then we denote $\Phi_\sigma(y,x)$ by $\Phi_\sigma^{\ +}(y,x)$. We use the following inequality.

$$-\frac{\pi}{2} < -\frac{\tau h}{4} < \tau\left(y_m - \frac{h}{2}\right) < \frac{\tau h}{4} < \frac{\pi}{2},$$

Then the following inequality holds.

$$\cos\tau\left(y_m - \frac{h}{4}\right) \ge \delta_0, y \in G_1 \cup \partial G_1, \text{ (22)}$$

Using inequality (22) and estimating the function $\Phi_\sigma(y,x)$ and its first-order derivatives, we obtain the following asymptotic estimate when $y \to \infty$.

$$\left|\Phi_\sigma(y,x)\right| + \left(\frac{\partial \Phi_\sigma(y,x)}{\partial x_i}\right) = O\left(\exp\left[-\delta_0 ch\tau\left|y'\right|\right]\right), \rho < \tau < 2\rho$$

. If $u(y) \in C^1(G_1) \cap C\left(\overline{G_1}\right)$ satisfies the systems (1) in domain G_1 have the growth conditions for the function $u(y)$ and its first-order derivatives.

$$\left|u(y)\right| + \left|\frac{\partial u(y)}{\partial y_i}\right| \le C\exp\left[\left(\exp 2\rho - \varepsilon\left|y'\right|\right)\right], \left|y'\right|^2 = y_1^{\ 2} + y_2^{\ 2} + \ldots y_{m-1}^{\ \ 2}, y \in G, \varepsilon > 0$$

Choosing in (21), $2\rho - \varepsilon < \tau < 2\rho$ then by the proven Theorem 1, we obtain the following integral formula.

$$u(x) = \int_{\partial G_1} M_\sigma(x,y) u(y) ds_y, x \in G_1, \text{ (23)}$$

Where $M_\sigma(x,y) = \left(E\left(\Phi_\sigma^{\ +}(y,x)\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D\left(t^T\right)$ If $u(y) \in C^1(G_2) \cap C\left(\overline{G_2}\right)$ satisfies the systems (1) in domain G_2 have the growth conditions for the function $u(y)$ and its first-order derivatives.

$$\left|u(y)\right| + \left|\frac{\partial u(y)}{\partial y_i}\right| \le C\exp\left[\left(\exp 2\rho - \varepsilon\left|y'\right|\right)\right], \left|y'\right|^2 = y_1^{\ 2} + y_2^{\ 2} + \ldots y_{m-1}^{\ \ 2}, y \in G, \varepsilon > 0$$

Then the following integral formula holds

$$u(x) = \int_{\partial G_2} M_\sigma(x,y) u(y) ds_y, x \in G_2 \text{ .(24)}$$

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Where $M_\sigma(x,y)=\left(E\left(\Phi_\sigma^-(y,x)\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D\left(t^T\right)$

Here $\Phi_\sigma^-(y,x)$ is defined by formulas (13), (14), and (15), where $K(w)$ is replaced by $K_2(w)$, the expression defined by the equation.

$$K_2(w)=K(w)\exp\left[-\delta chi\tau(w-h_1)-\delta_1 cpi\left(w-\frac{h}{2}\right)\right],$$

$$h_1=\frac{h}{2}+\frac{h}{4},\delta_1>0,\delta>0$$

IF $u(y)\in C^1(G_2)\cap C\left(\overline{G_2}\right)$ satisfies the systems (1) in domain G_2, have the growth conditions for the function

$$|u(y)|\le C\exp\left[a\cos\rho_1\left(y_m-\frac{h}{2}\right)\exp\rho_1\left|y'\right|\right],\left|y'\right|^2=y_1^2+y_2^2+...y_{m-1}^2,y\in\partial G,a\ge 0,\ ((25)$$

Then the integral formula of the form (24) holds.

Under the presence of boundary growth conditions, the integrals (23) and (24) converge absolutely and uniformly with respect to the parameter $\delta\ge 0$ for all values. $x\in G$ и $\sigma>0$. Let us put $\delta=0$ combine the integral formulas to obtain.

$$u(x)=\int_{\partial G}M_\sigma(x,y)u(y)ds_y,x\in G,\ (25),\qquad \text{where}$$

$$M_\sigma(x,y)=\left(E\left(\Phi_\sigma(y,x)\right)D^*\left(\frac{\partial}{\partial x}\right)\right)D\left(t^T\right)$$

Here $\Phi_\sigma(y,x)$ is defined by the formula.

$$\Phi_\sigma(y,x)=\left(\Phi^+{}_\sigma(y,x)\right)_{\delta=0}+\left(\Phi^-{}_\sigma(y,x)\right)_{\delta=0}$$

Functions $\Phi_\sigma(y,x)$ are defined by the formulas(13), (14), (15), where $K(w)$ is defined by (A), where it is assumed that $\delta=0$ The integral formula holds for any $\delta_1\ge 0$. By assuming $\delta_1=0$ we obtain the proof of the theorem.

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